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Coulomb System Equivalent to the Energy Spectrum of the Calogero–Sutherland–Moser (CSM) Model

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The purpose of this paper is to prove an equivalence between the energy spectrum of the CSM model and the electrostatic energy of a one-dimensional lattice of quantized point charges interacting via Coulomb potential with Dirichlet boundary conditions.

KEY WORDS: Coulomb system; CSM model energy spectrum.

1. INTRODUCTION

The CSM model is a one-dimensional soluble system of N particles interacting pairwise via Calogero's inverse quadratic potential[1] and put on a ring. Sutherland[2] introduced it first in the quantum case and solved the associated Schrödinger equation. Moser[3] and Ujino *et al.*[4] proved the complete integrability of the model in the classical and quantum case respectively. A systematic method to construct the periodic eigenfunctions of the system has been exposed by Sogo[5] while Forrester[6] has shown, among other results concerning groundstate correlation functions, that these solutions can be expressed as Jack polynomials. Current investigations e.g., by Lesage *et al.*[7] and by Serban *et al.*[8] concern dynamical correlation functions calculated for specific values of the coupling parameter, both integer[7] and rational ones,[8] the latter providing good models for studying fractional statistics.

This paper is dedicated to B. Jancovici in honor of his 65th birthday.

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In our work we have been primarily interested in the case of repulsive interactions although attractive ones are, to the extent discussed by Sutherland[2] also admissible. The potential energy being a convex function of the inter-particle separations in our case, we have investigated the Lattice Dynamics of the model and looked for an interpretation of its $N-2$ non trivial constants of motion in the limit of small excitation energies.[9] We have also been able to exhibit the occurence of "phonon energies" in the strong coupling limit of Sutherland's energy spectrum.[10]

Born out of these findings we present here still another interpretation of the energy spectrum, namely that of the electrostatic energy of a one-dimensional lattice of length N with quantized point charges interacting via the one dimensional Coulomb potential subject to Dirichlet boundary conditions. An advantage of this interpretation is that it is also valid in the case of attractive interactions. As to the quantum statistical aspects of the model, they have sofar not been considered in our papers.

In Section 2 we recall some basic facts concerning the hamiltonian of the model, its eigenvalues and eigenfunctions and also concerning its Lattice Statics and Dynamics. Furthermore, we indicate how some of the results obtained in ref. 10 can be easily reproduced.

In Section 3 we present the electrostatic analog of Sutherland's energy spectrum which, we believe, will shed some new light on the physics of the CSM model.

2. THE CSM MODEL

With x_j, $j=1,\dots, N$ standing for the coordinate of the jth particle of mass $m=1$ moving on a ring of circumference $=\pi$, with $\hbar=1$ and $(1/i)(\partial/\partial x_j)$ denoting the corresponding momentum operator and with $g^2(\sin x)^{-2}=g^2\sum_{n\in z}(x+n\pi)^{-2}$ being the periodized Calogero potential, the hamiltonian operator becomes

$$H=-\frac{1}{2}\sum_{1\leqslant j\leqslant N}\frac{\partial^2}{\partial x_j^2}+\sum_{1\leqslant i<j\leqslant N}\frac{g^2}{\sin^2(x_i-x_j)}. \tag{1}$$

It is convenient to recall here some facts concerning the Lattice Statics and Dynamics of the model. We disregard here the translational energy of the system. Let

$$x_j^0=-\frac{\pi}{2}+\left(j-\frac{1}{2}\right)\frac{\pi}{N}\qquad j=1,\dots, N \tag{2}$$

be the reference and equilibrium position of the jth particle of an order configuration such that $-\pi/2 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \leqslant \pi/2$ and let $u_j = x_j - x_j^0$ be the displacement from this position. The hamiltonian becomes:

$$H = -\frac{1}{2} \sum_{1 \leqslant j \leqslant N} \frac{\partial^2}{\partial u_j^2} + \sum_{1 \leqslant i < j \leqslant N} \frac{g^2}{\sin^2\left(\frac{\pi}{N} + u_i - u_j\right)}. \tag{3}$$

For the remainder of this paper we need only to know the static potential energy and the hamiltonian in the harmonic approximation. For the static potential energy we have[9]

$$V_0 = \frac{N}{2} \sum_{1 \leqslant t \leqslant N-1} \frac{g^2}{\sin^2\left(\frac{\pi t}{N}\right)} = g^2 \frac{N(N^2-1)}{6}. \tag{4}$$

In the harmonic approximation we have, for the square of the angular frequency ω_n,

$$\omega_n^2 = g^2 4 \sum_{1 \leqslant t \leqslant N-1} \sin^2 \frac{\pi n t}{N} \frac{1 + 2\cos^2 \frac{\pi t}{N}}{\sin^4 \frac{\pi t}{N}} = g^2 4 n^2 (N-n)^2 \tag{5}$$

i.e.

$$\omega_n = g\, 2n(N-n) \qquad n = 1, \dots, N-1. \tag{6}$$

Introducing the standard phonon raising and lowering operators a_n^+, a_n through

$$\begin{pmatrix} u_j \\ \dfrac{\partial}{\partial u_j} \end{pmatrix} = \frac{1}{\sqrt{N}} \sum_{1 \leqslant n \leqslant N-1} \exp\left(\frac{2i\pi j n}{N}\right) \begin{pmatrix} \sqrt{\dfrac{1}{2\omega_n}} (a_n + a_{N-n}^+) \\ \sqrt{\dfrac{1}{2}} \omega_n (a_n - a_{N-n}^+ \end{pmatrix} \tag{7}$$

results in

$$H_h = \sum_{1 \leqslant n \leqslant N-1} g\, 2n(N-n) \left(\frac{1}{2} + a_n^+ a_n\right). \tag{8}$$

At this point we remark that the N dependance of the zero point energy, namely

$$\varepsilon_{ph,0} = \sum_{1 \leqslant n \leqslant N-1} gn(N-n) = g\,\frac{N(N^2-1)}{6} \tag{9}$$

is the same as that of V_0 and also, as we shall see shortly, the same as that of the groundstate energy of the CSM model.

Let us come back to the original model and let $E(\mathbf{k})$ and $\psi_{\mathbf{k}}$ ($\mathbf{k}$ defined below) be the eigenvalues and the eigenfunctions of $H\psi_{\mathbf{k}} = E(\mathbf{k})\,\psi_{\mathbf{k}}$ with the periodic boundary conditions

$$\psi_{\mathbf{k}}(x_1,..., x_j+\pi,..., x_N) = \psi_{\mathbf{k}}(x_1,..., x_j,..., x_N). \tag{10}$$

The groundstate function

$$\psi_0 = \prod_{1 \leqslant i < j \leqslant N} |\sin(x_i - x_j)|^{\lambda} \tag{11}$$

is a solution of the Schrödinger equation if $\lambda(\lambda-1) = g^2$ with the positive root only $\lambda = 1/2 + (1/4 + g^2)^{1/2}$ being acceptable. The corresponding groundstate energy is

$$\mathbf{E}_0 = \lambda^2\,\frac{N(N^2-1)}{6}. \tag{12}$$

As pointed out in ref. 10, we observe that, with $\lambda^2 = g^2 + \lambda$, we have the remarkable identity

$$\mathbf{E}_0 = V_0 + \frac{\lambda}{g}\,\varepsilon_{ph,0}. \tag{13}$$

Consider now the excited states. There are indexed by sets of ordered quantum numbers $\mathbf{k} = (k_1,..., k_N)$ where $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_N$ and $k_j \in Z$, $j = 1,..., N$. The corresponding eigenfunctions are of the form $\psi_{\mathbf{k}} = \psi_0 \phi_{\mathbf{k}}$ and their construction as linear combinations of Laurent polynomials in the variables $\exp(2ik_j x_l)$ is given by Sogo.[5] Two different methods have been used by Sutherland[2] to obtain the eigenvalues $E(\mathbf{k})$ namely one, giving $E(\mathbf{k}) - E_0$ as diagonal elements of a triangular matrix and, the other one, in applying the asymptotic Bethe Ansatz. These eigenvalues read

$$E(\mathbf{k}) = E_0 + 2 \sum_{1 \leqslant i < j \leqslant N} \lambda(k_i - k_j) + 2 \sum_{1 < j \leqslant N} k_j^2 \tag{14}$$

$$= \lambda^2 \frac{N(N^2-1)}{6} + 2 \sum_{1 \leqslant j \leqslant N} \lambda(N+1-2j)\, k_j + 2 \sum_{1 \leqslant j \leqslant N} k_j^2 \tag{15}$$

$$= 2 \sum_{1 \leqslant j \leqslant N} \left(\lambda \left(\frac{N+1}{2} - j \right) + k_j \right)^2 \tag{16}$$

the last equality resulting from a non-trivial identity which is that

$$2 \sum_{1 \leqslant j \leqslant N} \left(\frac{N+1}{2} - j \right)^2 = 2 \sum_{1 \leqslant j \leqslant N} \left(\left(\frac{N+1}{2} \right)^2 - (N+1)\, j + j^2 \right)$$

$$= N(N+1) \left(\frac{N+1}{2} - (N+1) + \frac{2N+1}{3} \right)$$

$$= \frac{N(N^2-1)}{6} \tag{17}$$

and which proves that the two methods give the same result.

It seems appropriate to make contact here with the content of ref. 10 in showing that the identity[13] can be extended to all terms of $E(\mathbf{k})$ linear in λ. To do so we recall that new quantum numbers had been introduced through

$$k_j - k_{j+1} = v_i \geqslant 0 \qquad j = 1, \ldots, N-1 \tag{18}$$

and also the total momentum $K = \Sigma k_i$. Then it turned out that

$$2 \sum_{1 \leqslant j \leqslant N} \lambda(N+1-2f)\, k_j = \sum_{1 \leqslant n \leqslant N-1} \lambda\, 2n(N-n)\, v_n. \tag{19}$$

This result proved the conjecture that in the strong coupling limit where $\lambda = g + 0(1)$, the CSM energy spectrum is dominated by $V_0 + \varepsilon_{ph}$ and that, in general, the terms of $E(\mathbf{k})$ which are quadratic and linear in λ are reproduced by $V_0 + \lambda \varepsilon_{ph}/g$. It is in calculating the remaining purely kinetic energy terms of $E(\mathbf{k})$ which became a symmetric bilinear form in the quantum numbers v_n that the kernel discussed in the next section was found.

3. AN ELECTROSTATIC ANALOG

Let us introduce the new variables

$$p_j = \lambda \left(\frac{N+1}{2} - 1 \right) + k_i - \frac{P}{N} \qquad j = 1, \ldots, N \tag{20}$$

and the total momentum K through

$$P=\sum_{1\leqslant j\leqslant N}\lambda\left(\frac{N+1}{2}-j\right)+k_j=\sum_{1\leqslant j\leqslant N}k_i=K. \tag{21}$$

The energy spectrum becomes accordingly

$$E(K,\mathbf{p})=2\frac{K^2}{N}+2\sum_{1\leqslant i\leqslant N}p_j^2. \tag{22}$$

The idea is now to consider p_j as an electric field acting on the jth site of a one-dimensional lattice of length N. If so there exist an electrostatic potential, say φ_j, with p_j as it discretized gradient and also charges which are the discretized divergence of this electric field. Thus we set

$$\varphi_j-\varphi_j=p_j \qquad j=1,\dots,N. \tag{23}$$

Since $\sum_{1\leqslant j\leqslant N}p_j=0$, we have $\varphi_N-\varphi_0=0$ and, without loss of generality, we can choose $\varphi_N=\varphi_0=0$. This corresponds to Dirichlet boundary conditions. We have next the discretized Poisson equations

$$\begin{aligned}2\varphi_1-\varphi_2&=p_1-p_2=\lambda+k_1-k_2\\ -\varphi_{j-1}+2\varphi_j-\varphi_{j+1}&=p_j-p_{j-1}=\lambda+k_j-k_{j+1} \qquad j=2,\dots,N-2\\ -\varphi_{N-2}+2\varphi_{N-1}&=P_{N-1}-P_N=\lambda+k_{N-1}-K_N\end{aligned} \tag{24}$$

and the "charges"

$$\rho_j=\lambda+k_j-k_{j+1} \qquad i=1,\dots,N-1 \tag{25}$$

are >0 since $\lambda>0$. We recognize here the non-negative quantum number v_j introduced in ref. 10 namely $k_j-k_{j+1}=v_i$ for $j=1,\dots,N-1$. Now, electrostatics tells us that, with Δ_{ij} being the discretized Laplace operator with Dirichlet boundary conditions which can be red from Eq. 24, we have the three equivalent forms

$$\sum_{1\leqslant i\leqslant N}p_j^2=-\sum_{1\leqslant i,j\leqslant N}\Delta_{ij}\varphi_i\varphi_j=\sum_{1\leqslant m,n\leqslant N}G_{mn}\rho_m\rho_n \tag{26}$$

where

$$G_{mn}=(\Delta^{-1})_{mn}=\mathrm{Min}(m,n)-\frac{mn}{N}=\sum_{1\leqslant t\leqslant N-1}\frac{1}{N-1}\frac{\sin\left(\dfrac{m\pi t}{N}\right)\sin\left(\dfrac{n\pi t}{N}\right)}{4\sin^2(\pi t/2N)} \tag{27}$$

is the kernel or Green function of the one-dimensional Coulomb potential with Dirichlet boundary conditions and the quantized "charges" ρ_n are given

$$\rho_n = \lambda + \nu_n \qquad n = 1, \ldots, N-1. \tag{28}$$

The result is consequently that

$$E(\mathbf{k}) = E(K, \nu) = 2\frac{K^2}{N} + 2 \sum_{1 \leqslant m, n \leqslant N-1} G_{mn}(\lambda + \nu_m)(\lambda + \nu_n). \tag{29}$$

It remains to check that this equality holds. We have firstly that the purely kinetic terms are indeed those of Eq. (53) in ref. 10, secondly that the linear term in λ, namely

$$4\lambda \sum_{1 \leqslant m, n \leqslant N-1} G_{mn} \nu_n$$

become with

$$\begin{aligned}
4 \sum_{1 \leqslant m \leqslant N-1} G_{mn} &= 4 \sum_{1 \leqslant m \leqslant N-1} \left(\min(m, n) - \frac{mn}{N} \right) \\
&= 4 \left(\sum_{1 \leqslant m \leqslant n} m + \sum_{n+1 \leqslant m \leqslant N-1} m - \sum_{1 \leqslant m \leqslant N-1} m \frac{mn}{N} \right) \\
&= 2n(n+1) + 4n(N-1-n) - 2n(N-n) \\
&= 2m(N-n),
\end{aligned}$$

i.e.,

$$4\lambda \sum_{1 \leqslant m, n \leqslant N-1} G_{mn} \nu_n = \lambda \sum_n 2n(N-n)\, \nu_n = 2\lambda \sum_{1 \leqslant j \leqslant N-1} (N+1-2j)\, k_j$$

according to Eq. (23). Thirdly and obviously we have that

$$2 \sum_{1 \leqslant m, n \leqslant N-1} G_{mn} = \sum_{1 \leqslant n \leqslant N-1} m(N-m) = \tfrac{1}{6} N(N^2 - 1)$$

according to Eq. (9).

The analysis presented in ref. 10 and in this paper have revealed a few remarkable identities. We believe that they are not fortuitous and that a deeper understanding of their origin will shed new light on some interesting physical and mathematical properties of the CSM model.

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